Chapter 1. Electrostatics I

Notes:

- *Most of the material presented in this chapter is taken from Jackson, Chap. 1.*
- *Units from the Système International (SI) will be used in this chapter.*

1.1 Mathematical Considerations

1.1.1 Dirac's delta function (distribution)

a) Dirac's delta function is defined such that

$$\delta(x-a)=\begin{cases}\infty, & \text{for } x=a\\ 0, & \text{for } x\neq a\end{cases} \tag{1.1}$$

but with the restriction that the "area" it encloses equals 1. Therefore, whenever it is used it is implied that

$$\int_{-\infty}^{\infty}\delta(x-a)dx=1. \tag{1.2}$$

b) It is also required from the delta function that for any good function $g(x)$ (i.e., one that is differentiable everywhere any number of times, while it and its derivatives vanish faster than any power of $1/|x|$ as $|x|\to\infty$) we have

$$\int_{-\infty}^{\infty}g(x)\delta(x-a)dx=g(a). \tag{1.3}$$

Dirac's function can be approximated with many different functions. For example, the function

$$D_n(x)=\sqrt{\frac{n}{\pi}}e^{-nx^2}\qquad n=1,2,\ldots \tag{1.4}$$

can be shown to satisfy equation (1.3) when $n\to\infty$. To prove this, we first set $y=x-a$ in the integrand of equation (1.3), and we replace $\delta(y)$ by $D_n(y)$

$$\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}g(y+a)e^{-ny^2}dy=g(a). \tag{1.5}$$

We next evaluate the following (with $|\cdots|$ denoting the absolute value)

$$\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} g(y+a)e^{-ny^2}dy - g(a)\right| = \sqrt{\frac{n}{\pi}}\left|\int_{-\infty}^{\infty} e^{-ny^2}\left[g(y+a)-g(a)\right]dy\right|, \tag{1.6}$$

since

$$\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-ny^2}dy = 1. \tag{1.7}$$

Because it is always the case that

$$\left|g(y+a)-g(a)\right| \le |y|\cdot\max\left\{\left|\frac{dg}{dy}\right|\right\}, \tag{1.8}$$

then,

$$\begin{aligned}\sqrt{\frac{n}{\pi}}\left|\int_{-\infty}^{\infty} e^{-ny^2}\left[g(y+a)-g(a)\right]dy\right| &\le \max\left\{\left|\frac{dg}{dy}\right|\right\}\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}|y|e^{-ny^2}dy \\ &\le \frac{1}{\sqrt{n\pi}}\max\left\{\left|\frac{dg}{dy}\right|\right\}\underset{n\to\infty}{\to} 0.\end{aligned} \tag{1.9}$$

Equation (1.3) is, therefore, verified with any good function $g(x)$ for $D_\infty(x)$.

c) The derivative of the delta function can also be defined and used as follows

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\delta'(x-a)dx &= g(x)\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(x)\delta(x-a)dx \\ &= -g'(a),\end{aligned} \tag{1.10}$$

where we integrated by parts and $g' = dg/dx$.

d) Since from equation (1.1) $\delta(x)$ is zero everywhere except at $x = 0$, we can write

$$\int_{-\infty}^{\infty}\delta(x-a)dx = \int_{a-\varepsilon}^{a+\varepsilon}\delta(x-a)dx, \tag{1.11}$$

with $\varepsilon > 0$. Then for a function $y(x)$ with multiple zeros

$$\int_{-\infty}^{\infty} g(x)\delta\left[y(x)\right]dx = \sum_i \int_{y(x_i)-\varepsilon}^{y(x_i)+\varepsilon} g\left[x(y)\right]\delta(y)\frac{dy}{\left|\frac{dy}{dx}\right|}, \tag{1.12}$$

since $dx = \frac{dx}{dy}dy$. We can, therefore, write

$$\delta\big[y(x)\big]=\sum_i \frac{\delta(x-x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}}, \tag{1.13}$$

where the x_i's are the roots of the equation $y(x)=0$.

e) Dirac's distribution is the derivative of the so-called **Heaviside step function** $H(x)$ (also a distribution)

$$H(x)=\begin{cases}1, & \text{for } x>0\\ 0, & \text{for } x<0\end{cases} \tag{1.14}$$

For a good function $g(x)$

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dH(x)}{dx}g(x)dx &= H(x)g(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}H(x)\frac{dg(x)}{dx}dx\\ &=-\int_0^{\infty}\frac{dg(x)}{dx}dx=g(0).\end{aligned} \tag{1.15}$$

From equations (1.3) and (1.15) we see that

$$\delta(x)=\frac{dH(x)}{dx}. \tag{1.16}$$

f) Finally, higher-dimensional versions of the delta function are the product of delta functions from each dimension. For example, in three-dimension

$$\delta(\mathbf{x}-\mathbf{X})=\delta(x_1-X_1)\delta(x_2-X_2)\delta(x_3-X_3) \tag{1.17}$$

vanishes everywhere except at $\mathbf{x}=\mathbf{X}$, and

$$\int_{\Delta V}\delta(\mathbf{x}-\mathbf{X})d^3x=\begin{cases}1, & \text{if } \Delta V \text{ contains } \mathbf{x}=\mathbf{X}\\ 0, & \text{if } \Delta V \text{ does not contain } \mathbf{x}=\mathbf{X}\end{cases} \tag{1.18}$$

1.1.2 **Tensor notation**

The **Kronecker tensor** is defined by

$$\delta_{jk} = \begin{cases} 1, & \text{for } j = k \\ 0, & \text{for } j \neq k \end{cases} \tag{1.19}$$

where j and k are tensor indices (e.g., in a three-dimensional space the indices can take the value 1, 2, or 3). This tensor is useful to compactly express some vector expressions. For example, the dot product of two vectors can be written as

$$\mathbf{A} \cdot \mathbf{B} = \delta_{jk} A_k B_j = A_j B_j, \tag{1.20}$$

where a summation on repeated indices is implied (i.e., $A_j B_j \equiv \sum_j A_j B_j$).

The **Levi-Civita tensor** is defined by

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation of } ijk \\ -1, & \text{odd permutation of } ijk \\ 0, & \text{if } i = j,\ i = k, \text{ or } j = k \end{cases} \tag{1.21}$$

For example, $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$, and $\varepsilon_{113} = \varepsilon_{121} = \varepsilon_{133} = \cdots = 0$. This tensor can be used to compactly express the cross product

$$\left[\mathbf{A} \times \mathbf{B}\right]_i = \varepsilon_{ijk} A_j B_k. \tag{1.22}$$

There also exist many relations that link the Kronecker and Levi-Civita tensors. Most notably,

$$\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}. \tag{1.23}$$

As examples of the usefulness of these tensors, consider the following vector formulas

$$\begin{aligned} \left[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\right]_i &= \varepsilon_{ijk} A_j \varepsilon_{kmn} B_m C_n \\ &= \varepsilon_{kij} \varepsilon_{kmn} A_j B_m C_n \\ &= \left(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}\right) A_j B_m C_n \\ &= A_j C_j B_i - A_j B_j C_i \\ &= (\mathbf{A} \cdot \mathbf{C}) B_i - (\mathbf{A} \cdot \mathbf{B}) C_i, \end{aligned} \tag{1.24}$$

or,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}, \tag{1.25}$$

and with ϕ and ψ two scalar functions

$$\begin{aligned}\nabla\cdot(\phi\nabla\psi)&=\partial_i(\phi\partial_i\psi)\\&=(\partial_i\phi)(\partial_i\psi)+\phi\partial_i\partial_i\psi\\&=(\nabla\phi)\cdot(\nabla\psi)+\phi\nabla^2\psi,\end{aligned} \tag{1.26}$$

where $\partial_i\equiv\dfrac{\partial}{\partial x_i}$.

1.1.3 **The divergence theorem**

Consider an infinitesimal volume Δ^3x bounded by the surface of a cube whose sides have an area Δa (see Figure 1.1). Defining $\mathbf{n}_i$ as a unit vector pointing out of the volume element (with $i=1,2,3$ or x,y,z) and therefore normal to the two (parallel) surfaces that are perpendicular to the x_i-axis, we are interested in calculating the following quantity

$$\sum_{\substack{\text{over six}\\ \text{surfaces}}}\mathbf{D}\cdot\mathbf{n}_i\,\Delta a, \tag{1.27}$$

where $\mathbf{D}$ is some arbitrary vector, and the summation is taken over the six sides of the cube. Concentrating first on the surfaces corresponding to $i=1$ or x, and setting the value of the corresponding component of $\mathbf{D}$ at the "back" surface of Figure 1.1 to D_x, we can approximate that at the "front" surface (located a distance Δx ahead of the previous one) we will have $D_x+\partial D_x/\partial x\,\Delta x$ for the same component of $\mathbf{D}$. We can then write the corresponding contribution to equation (1.27) as

$$\left[\left(D_x+\frac{\partial D_x}{\partial x}\Delta x\right)-D_x\right]\Delta y\Delta z=\frac{\partial D_x}{\partial x}\Delta x\Delta y\Delta z=\frac{\partial D_x}{\partial x}\Delta^3x, \tag{1.28}$$

since the unit normal vector to the front (back) surface is along $\mathbf{e}_x$ ($-\mathbf{e}_x$). Using a similar procedure for the other four surfaces, it can easily be shown that

$$\sum_{\substack{\text{over six}\\ \text{surfaces}}}\mathbf{D}\cdot\mathbf{n}_i\,\Delta a=(\nabla\cdot\mathbf{D})\Delta^3x. \tag{1.29}$$

We can extend this treatment to an arbitrary large volume V provided that we divide it into an ensemble of adjacent infinitesimal cubes. If we further denote by S the closed surface that contains V, we can finally write the so-called **divergence theorem** (sometimes called **Gauss' theorem**) as

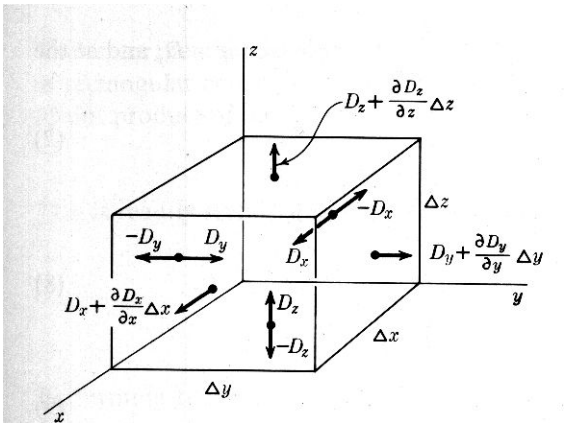

Figure 1.1 – Illustration of the divergence theorem.

$$\boxed{\int_V \nabla \cdot \mathbf{D}\, d^3x = \int_S \mathbf{D} \cdot \mathbf{n}\, da} \tag{1.30}$$

The generalization from equation (1.29) to equation (1.30) arises from the fact that the sides of every infinitesimal cube that is *interior* to the volume V will cancel with the side of an adjacent cube, leaving only the *exterior* side of the cubes that make out the closed surface S.

Similarly, if we consider the components of $\mathbf{D}$ which are parallel to the surfaces of the infinitesimal cubes (i.e., $\mathbf{n}_i \times \mathbf{D}$), it can similarly be shown that

$$\boxed{\int_V \nabla \times \mathbf{D}\, d^3x = \int_S \mathbf{n} \times \mathbf{D}\, da} \tag{1.31}$$

Finally, we could apply the same procedure to a scalar function ψ to get

$$\boxed{\int_V \nabla \psi\, d^3x = \int_S \psi \mathbf{n}\, da} \tag{1.32}$$

1.1.4 **Stokes' theorem**

Let's consider an open surface S with an exterior right-handed contour C that we divide into a large number of infinitesimal square loops (see Figure 1.2). We are interested in evaluating the following quantity

$$\sum_{\substack{\text{over four} \\ \text{segments}}} \mathbf{E} \cdot d\mathbf{l}, \tag{1.33}$$

where $\mathbf{E}$ is some vector, and $\mathbf{l}$ the right-handed contour of the infinitesimal loop (made up with four segment vectors) over which the summation is taken. If we denote the

"bottom" (top) segment vector by $\mathbf{a}$ $(-\mathbf{a})$ and the "right" (left) one by $\mathbf{b}$ $(-\mathbf{b})$, we can approximate equation (1.33) with

$$\begin{aligned}\sum_{\substack{\text{over four}\\ \text{segments}}} \mathbf{E}\cdot d\mathbf{l} &= \underset{\text{bottom}}{\mathbf{E}\cdot\mathbf{a}} - \underset{\text{top}}{\left[\mathbf{E}+(\mathbf{b}\cdot\nabla)\mathbf{E}\right]}\cdot\mathbf{a} + \underset{\text{right}}{\left[\mathbf{E}+(\mathbf{a}\cdot\nabla)\mathbf{E}\right]}\cdot\mathbf{b} - \underset{\text{left}}{\mathbf{E}\cdot\mathbf{b}} \\ &= \mathbf{b}\cdot\left[(\mathbf{a}\cdot\nabla)\mathbf{E}\right] - \mathbf{a}\cdot\left[(\mathbf{b}\cdot\nabla)\mathbf{E}\right] \\ &= b_k a_j \partial_j E_k - a_j b_k \partial_k E_j \\ &= a_j b_k \left(\partial_j E_k - \partial_k E_j\right) \\ &= a_j b_k \partial_m E_n \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) \\ &= \left(\varepsilon_{ijk} a_j b_k\right)\left(\varepsilon_{imn}\partial_m E_n\right) \\ &= (\mathbf{a}\times\mathbf{b})\cdot(\nabla\times\mathbf{E}),\end{aligned} \tag{1.34}$$

where we have resorted to the tensor notation and used equation (1.23). If we further define $\mathbf{n}$ as the unit vector normal to the loop's surface and Δa as the loop's area, we have

$$\mathbf{a}\times\mathbf{b} = \mathbf{n}\,\Delta a. \tag{1.35}$$

Inserting equation (1.35) into equation (1.34), and combining it with the results obtained for all the other loops (included in S) using this same procedure, we can write the so-called **Stokes' theorem**

$$\boxed{\int_S (\nabla\times\mathbf{E})\cdot\mathbf{n}\,da = \oint_C \mathbf{E}\cdot d\mathbf{l}} \tag{1.36}$$

The generalization of the results obtained for one loop to the whole surface S is possible because of the fact that contributions along common segments of adjacent infinitesimal loops that are *interior* to the surface S will cancel each other, leaving only contributions along the *exterior* sides of the loops that make out the closed circuit C.

Finally, applying the same technique to the case of a scalar function ψ will yield the following relation

$$\boxed{\int_S \mathbf{n}\times\nabla\psi\,da = \oint_C \psi\,d\mathbf{l}} \tag{1.37}$$

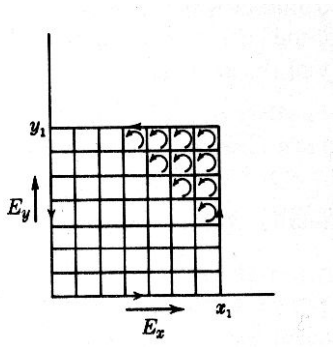

Figure 1.2 – Illustration of Stokes' theorem.

1.2 Coulomb's Law

Coulomb's Law states that *the force acting between two charged bodies at rest with respect with each other is inversely proportional to the square of the distance between them, proportional to each charge, directed along the line joining them, and repulsive for like charges and attractive for opposite charges*.

Mathematically speaking the force defined in Coulomb's Law is expressed as follows

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\mathbf{x}_1 - \mathbf{x}_2}{\left|\mathbf{x}_1 - \mathbf{x}_2\right|^3}, \tag{1.38}$$

where $\mathbf{x}_i$ is the position of the charge q_i, and $\left(4\pi\varepsilon_0\right)^{-1} = 10^{-7} c^2$ so that the permittivity of vacuum $\varepsilon_0 \simeq 8.854 \times 10^{-12}$ (F/m) (c is the speed of light in vacuum, and F stands for "Farad"). The SI unit for charge is the *coulomb* (C); the electron's charge is $q \simeq 1.602 \times 10^{-19}$ C.

The **electric field** is defined as the force per unit charge at a given point. That is,

$$\mathbf{F} = q\mathbf{E}, \tag{1.39}$$

with $\mathbf{E}$ the electric field (which has units of Volts per meter (V/m)). Combining equation (1.38) and (1.39), it is seen that a charge q_1 generates an electric field, which according to Coulomb's Law is given by

$$\mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}_1}{\left|\mathbf{x} - \mathbf{x}_1\right|^3}. \tag{1.40}$$

When an aggregate of charge charges is involved, the field generated is simply calculated from the linear superposition of the field generated by each charge

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \frac{\mathbf{x} - \mathbf{x}_1}{\left|\mathbf{x} - \mathbf{x}_1\right|^3}. \tag{1.41}$$

Alternatively, in the continuous limit where the charge density $\rho(\mathbf{x})$ (i.e., the amount of charge per unit volume) is given, then

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'. \tag{1.42}$$

1.3 Gauss' Law and the Electric Field

We consider the electric field generated by a single point charge q that is enclosed within an arbitrary surface S. Let's denote by da an infinitesimal surface element at a given point on the surface, and located a distance r away from the charge. If the electric field at that point makes an angle θ with the normal vector $\mathbf{n}$ to da, then we can write from equation (1.40)

$$\begin{aligned}\mathbf{E}\cdot\mathbf{n}\, da &= \frac{q}{4\pi\varepsilon_0}\frac{\cos(\theta)}{r^2}\, da \\ &= \frac{q}{4\pi\varepsilon_0} d\Omega,\end{aligned} \tag{1.43}$$

since the solid angle subtended by da at the charge is given by $r^2 d\Omega = \cos(\theta) da$. If we now integrate over the whole surface we get

$$\int_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{q}{\varepsilon_0}. \tag{1.44}$$

It is important to realize that *if the charge were located outside of the volume V, then the right-hand side of equation (1.44) would be zero*. Equation (1.44) can be generalized to cases involving multiple charges

$$\int_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{1}{\varepsilon_0}\sum_i q_i, \tag{1.45}$$

or a charge density

$$\boxed{\int_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x}')\, d^3x'} \tag{1.46}$$

where V is the volume enclosed by S. The right-hand side of equation (1.46) represents the total charge contained in V. Equation (1.46) is **Gauss' Law** for a charge density.

We can transform this last equation using the divergence theorem (i.e., equation (1.30)) to get

$$\int_V \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} \right) d^3x' = 0. \tag{1.47}$$

But since the volume is arbitrary, equation (1.47) implies that

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}} \tag{1.48}$$

which is the differential form of Gauss' Law of electrostatics.

1.4 The Scalar Potential Function

We now go back to equation (1.42) for the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x', \tag{1.49}$$

and we note that part of the integrand can be simplified as follows

$$\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right). \tag{1.50}$$

Inserting equation (1.50) into equation (1.49) we find a new equation for the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{-1}{4\pi\varepsilon_0} \nabla \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'. \tag{1.51}$$

We are now in a position to introduce the **scalar potential** function $\Phi(\mathbf{x})$ (with units of Volts (V))

$$\boxed{\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'} \tag{1.52}$$

from which the electric field can be calculated with

$$\boxed{\mathbf{E}(\mathbf{x}) = -\nabla \Phi(\mathbf{x})} \tag{1.53}$$

Since it is always true that for any scalar function $\Phi(\mathbf{x})$ we have $\nabla \times \nabla \Phi = 0$, then it follows from equation (1.53) that

$$\boxed{\nabla \times \mathbf{E} = 0} \tag{1.54}$$

If we consider the work that is required to move a test charge q from one point to another in a given electric field, we have

$$\begin{aligned} W &= -\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}\cdot d\mathbf{l} = -q\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E}\cdot d\mathbf{l} \\ &= q\int_{\mathbf{x}_1}^{\mathbf{x}_2} \nabla\Phi\cdot d\mathbf{l} = q\int_{\mathbf{x}_1}^{\mathbf{x}_2} d\Phi \\ &= q\left[\Phi(\mathbf{x}_2)-\Phi(\mathbf{x}_1)\right]. \end{aligned} \tag{1.55}$$

We see from this last result that we can equate the quantity $q\Phi$ to the potential energy of the test charge.

Example

Use Gauss' Law and equation (1.54) to prove the following:

a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)

b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shields its exterior from the fields due to charges placed inside it.

c) The electric field at the surface of a conductor is normal to the surface and has a magnitude of σ/ε_0, where σ is the charge density per unit area on the surface (i.e., it is the surface charge density).

Solution.

a) Because of the given definition, the electric field inside a conductor must always be zero, since the sudden movement of the free moving charges within the conductor will quickly counterbalance any applied electric field. Now, let's define a closed surface S that encloses the conductor completely, as well as some small amount of volume exterior to the conductor. From Gauss' Law (i.e., equation (1.44)) we can write

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{\varepsilon_0}, \tag{1.56}$$

where q is the excess charge placed on the conductor. We further break the volume enclosed by S into two sub-volumes. The first one is delimited by the closed surface S_1, which is entirely contained within the conductor but does not include its boundary, while the other, delimited by a surface S_2, enclosing the boundary and the remainder of the volume within S that is exterior to the conductor. We can therefore write

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \int_{S_1} \mathbf{E}\cdot\mathbf{n}_1\,da_1 + \int_{S_2} \mathbf{E}\cdot\mathbf{n}_2\,da_2 = \frac{q}{\varepsilon_0}, \tag{1.57}$$

but since the electric field in the conductor is known to be zero, then equation (1.57) simplifies to

$$\int_{S_2} \mathbf{E}\cdot\mathbf{n}_2\,da_2 = \frac{q}{\varepsilon_0}. \tag{1.58}$$

It is now straightforward to see that through a limiting process we could modify the boundary of S_2 (and by the same token S_1) such that the volume it encloses is limited to the surface of the conductor. It follows that the excess charge has to be located on this surface.

b) We consider three volumes: one enclosing the cavity at the interior of the conductor (with a boundary S_1 that does not include the inner boundary of the conductor), a second (of boundary S_2) encloses entirely, and only, the conductor, and another (boundary S_3) enclosing the volume exterior to the conductor. If we first place a charge q_{int} within S_1 the free moving charges of the conductor will rearrange themselves such that the field in the conductor is zero. That is, charges of polarity opposite to q_{int} will migrate to the inner surface of the conductor in such a way that the electric field in the conductor is zero everywhere. Furthermore, because of Gauss' Law the total charge on the inner surface must equal $-q_{\text{int}}$. But since the conductor must remain globally neutral, it must be that a charge of $+q_{\text{int}}$ will migrate to the outer surface of the conductor. Then, by Gauss's Law the electric field in S_3 caused by the charges within S_1 and S_2 must obey the following relation

$$\int_{S_1} \mathbf{E}\cdot\mathbf{n}_1\,da_1 + \int_{S_2} \mathbf{E}\cdot\mathbf{n}_2\,da_2 = \frac{q_{\text{int}}}{\varepsilon_0}. \tag{1.59}$$

That is, the region external to the conductor is not shielded by charges located within the cavity.

On the other hand, if we place a charge q_{ext} in the volume contained in S_3, then free moving charges within the conductor will migrate to the outer surface of the conductor to ensure that the electric field is zero everywhere within the conductor. More precisely, charges of polarity opposite to q_{ext} will be located on a part of the outer surface of the conductor that is closer to the external charge (i.e., where $\cos(\theta) < 0$ in equation (1.43)), and charge of like polarity to q_{ext} will be on the rest of the outer surface (i.e., where $\cos(\theta) > 0$ in equation (1.43)). The important thing to realize is that the total charge on the outer surface of the conductor is equal to zero, and since the electric field within the conductor is also zero, then there will be no migration of charges to the inner surface of the conductor (caused by the presence of q_{ext} outside

the conductor). That is, the volume located within the outer boundary of the conductor is unaffected by q_{ext} owing to the rearrangement of charges on that boundary, and the cavity is shielded from charges external to the conductor.

c) We first refer to the infinitesimal pillbox of Figure 1.3 that straddles the outer boundary of the conductor with S. From Gauss' Law we know that

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{\varepsilon_0}, \tag{1.60}$$

where S is the closed surface delimiting the pillbox, $\mathbf{n}$ is the unit vector normal to S, and q is the total charge located inside the box. In the limit where we let the sides of the pillbox shrink to an infinitesimally small size, q is then completely located on the boundary, and we can simplify equation (1.60) to

$$\left(\mathbf{E}_\perp - \mathbf{E}_{\text{cond}}\right)\cdot\mathbf{n}\,\Delta a = \frac{q}{\varepsilon_0}, \tag{1.61}$$

with Δa the area of the top and bottom surface of the pillbox. We know, however, that the electric field $\mathbf{E}_{\text{cond}}$ inside the conductor is zero. We then find that the magnitude of the electric field $\mathbf{E}_\perp$ perpendicular to the surface of the connector is

$$E_\perp = \frac{\sigma}{\varepsilon_0}, \tag{1.62}$$

where $\sigma = q/\Delta a$ is the surface charge density. Finally, considering the infinitesimal loop of Figure 1.3, and letting their sides shrink to an infinitesimally small size, we find that the tangential electric field at the surface of the conductor is zero. We can ascertain this by integrating the electric field around the loop, and using Stokes' theorem with equation (1.54). More precisely,

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_A \nabla\times\mathbf{E}\,da = 0, \tag{1.63}$$

where A is the (open) surface covered by the loop. When the sides of the loop are infinitesimally small we have (remembering that electric field inside the conductor is zero)

$$\mathbf{E}_{\parallel}\cdot\Delta\mathbf{l} = 0. \tag{1.64}$$

The combination of equations (1.62) and (1.64) the fact the electric field at the surface of the conductor is perpendicular to the surface and has a magnitude of σ/ε_0.

1.5 The Equations of Laplace and Poisson

If we start with equation (1.48) for the divergence of the electric field (i.e., $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$) and we combine it with equation (1.53) relating $\mathbf{E}$ to the potential (i.e., $\mathbf{E} = -\nabla\Phi$), we get the so-called **Poisson equation**

$$\boxed{\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}} \tag{1.65}$$

In regions where charges are absent, we have the **Laplace equation**

$$\boxed{\nabla^2\Phi = 0} \tag{1.66}$$

Incidentally, the Poisson equation can be used to evaluate an important relation that connects the function $1/r$, which often appears in electrostatics, and the Dirac delta function. If we consider the scalar potential at a position $\mathbf{x}$ due to a single point charge q located at $\mathbf{x}''$, from equation (1.65) and $\rho(\mathbf{x}) = q\,\delta(\mathbf{x} - \mathbf{x}'')$ we get

$$\nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x} - \mathbf{x}''). \tag{1.67}$$

But we also know from equation (1.52) that

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 |\mathbf{x} - \mathbf{x}''|}. \tag{1.68}$$

Hence, it must be the case that

$$\boxed{\nabla^2\left(\frac{1}{|\mathbf{x} - \mathbf{x}''|}\right) = -4\pi\delta(\mathbf{x} - \mathbf{x}'')} \tag{1.69}$$

1.6 Boundary Conditions, and Single- and Double-layer Surface-charge Distributions

Let's consider the interface between two different media delimited by a surface S (with a unit normal vector $\mathbf{n}$ directed from medium 1 to medium 2) on which a surface-charge density $\sigma(\mathbf{x})$ is distributed. Let the volume V enclosed by the surface be that of a small pillbox (see Figure 1.3). Using equation (1.46) (i.e., Gauss' Law) we can write

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x}')\, d^3x'. \tag{1.70}$$

In the limit where the height of the pillbox is made infinitesimally small, so that its end surfaces (of area Δa) are just on either side of the boundary, we have

$$\left(\mathbf{E}_2 - \mathbf{E}_1\right) \cdot \mathbf{n}\, \Delta a = \frac{\sigma}{\varepsilon_0}\, \Delta a. \tag{1.71}$$

We therefore find that *there is a discontinuity in the normal component of the electric at the boundary between the two media, which is proportional to the surface-charge density*.

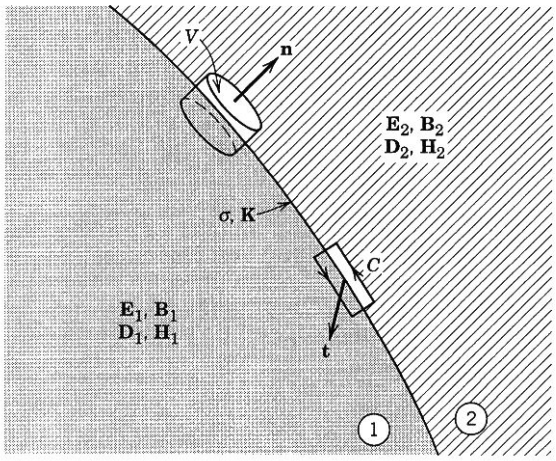

Figure 1.3 – The boundary surface between two media. We determine the boundary conditions for the electric field and the scalar potential by considering a small pillbox S and a small rectangular contour C.

Alternatively, we consider a rectangular loop C that straddles the boundary as shown in Figure 1.3 (with the vector $\mathbf{t}$ normal to both the plane of the loop and $\mathbf{n}$), and we let the segments parallel to $\mathbf{n}$ become infinitesimally short. We can write from equation (1.54) and Stokes' theorem (i.e., equation (1.36)) that

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \left(\mathbf{t} \times \mathbf{n}\right) \cdot \left(\mathbf{E}_2 - \mathbf{E}_1\right) \Delta l = 0, \tag{1.72}$$

with Δl the length of the segments of C that are parallel to the boundary. Since $\mathbf{a} \cdot \left(\mathbf{b} \times \mathbf{c}\right) = \mathbf{c} \cdot \left(\mathbf{a} \times \mathbf{b}\right)$, and the orientation of the vector $\mathbf{t}$ (and, therefore, the orientation of the contour C) is arbitrary on the surface of the boundary, we can transform equation (1.72) to

$$\mathbf{n} \times \left(\mathbf{E}_2 - \mathbf{E}_1\right) = 0. \tag{1.73}$$

That is, *the component of the electric field tangential to the boundary is continuous*.